Name: Solutions

Math 260
Exam 2

Please show ALL your work on the problems below. No more than 1 point will be given to problems if *you only provide the correct answer and insufficient work.*

A formula you may need: $\vec{v}\cdot\vec{w} = ||\vec{v}||\,||\vec{w}||\cos\theta$

1. (10, 5, 10 points) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -2 & -1 \\ 9 & 0 & -3 \end{bmatrix}$,

a) Find $\det(A)$ using the cofactor expansion method (NO calculator)

I used the 2nd row

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = -4\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 9 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 9 & 0 \end{vmatrix}$$

$$= -4\left[(1)(-3) - (0)(1)\right] - 2\left[(2)(-3) - (9)(1)\right] + 1\left[(2)(0) - (9)(1)\right]$$

$$= -4[-3] - 2[-15] + 1[-9]$$

$$= 12 + 30 - 9$$

$$= \boxed{33}$$

b) Is A invertible? Why or why not?

Yes, A is invertible bec. $|A| \neq 0$ $(|A| = 33)$

c) Find A^{-1} using the adjoint method (No calculator)

$$\operatorname{adj}(A) = \begin{bmatrix} +\begin{vmatrix} -2 & -1 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 9 & -3 \end{vmatrix} & +\begin{vmatrix} 4 & -2 \\ 9 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 9 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 9 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 6 & 3 & 18 \\ 3 & -15 & 9 \\ 1 & 6 & -8 \end{bmatrix}^T = \begin{bmatrix} 6 & 3 & 1 \\ 3 & -15 & 6 \\ 18 & 9 & -8 \end{bmatrix}$$

$$\text{So } A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{33}\begin{bmatrix} 6 & 3 & 1 \\ 3 & -15 & 6 \\ 18 & 9 & -8 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{11} & \frac{1}{11} & \frac{1}{33} \\ \frac{1}{11} & \frac{-5}{11} & \frac{2}{11} \\ \frac{6}{11} & \frac{3}{11} & -\frac{8}{33} \end{bmatrix}}$$

2. (12 points) Use Cramer's Rule to solve the system of equations below. (Calculator OK)

$$7x_1 + 2x_2 + x_3 = 19$$
$$-3x_1 + 4x_2 - 2x_3 = -4$$
$$2x_1 - 5x_2 + 6x_3 = 5$$

$$x_1 = \frac{\begin{vmatrix} 19 & 2 & 1 \\ -4 & 4 & -2 \\ 5 & -5 & 6 \end{vmatrix}}{\begin{vmatrix} 7 & 2 & 1 \\ -3 & 4 & -2 \\ 2 & -5 & 6 \end{vmatrix}} = \frac{294}{133} = \frac{42}{19}$$

$$x_2 = \frac{\begin{vmatrix} 7 & 19 & 1 \\ -3 & -4 & -2 \\ 2 & 5 & 6 \end{vmatrix}}{\begin{vmatrix} 7 & 2 & 1 \\ -3 & 4 & -2 \\ 2 & -5 & 6 \end{vmatrix}} = \frac{161}{133} = \frac{23}{19}$$

$$x_3 = \frac{\begin{vmatrix} 7 & 2 & 19 \\ -3 & 4 & -4 \\ 2 & -5 & 5 \end{vmatrix}}{\begin{vmatrix} 7 & 2 & 1 \\ -3 & 4 & -2 \\ 2 & -5 & 6 \end{vmatrix}} = \frac{147}{133} = \frac{21}{19}$$

$$\boxed{x_1 = \frac{42}{19}, \quad x_2 = \frac{23}{19}, \quad x_3 = \frac{21}{19}}$$

3. (15, 5, 15, 5 points) Let $A = \begin{bmatrix} 1 & -15 & -6 \\ 0 & -4 & -2 \\ 0 & 3 & 1 \end{bmatrix}$.

a) Find the characteristic polynomial of A (No calculator)

$$C_A(x) = |xI - A| = \left| x\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -15 & -6 \\ 0 & -4 & -2 \\ 0 & 3 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 1 & -15 & -6 \\ 0 & -4 & -2 \\ 0 & 3 & 1 \end{bmatrix} \right|$$

expanded along 1st column

$$= \begin{vmatrix} x-1 & 15 & 6 \\ 0 & x+4 & 2 \\ 0 & -3 & x-1 \end{vmatrix} = +(x-1)\begin{vmatrix} x+4 & 2 \\ -3 & x-1 \end{vmatrix} \underbrace{-0\begin{vmatrix} 15 & 6 \\ -3 & x-1 \end{vmatrix}}_{=0} \underbrace{+0\begin{vmatrix} 15 & 6 \\ x+4 & 2 \end{vmatrix}}_{=0}$$

$$= (x-1)\left[(x+4)(x-1) - (-3)(2)\right] = (x-1)\left[x^2+3x-4+6\right]$$

$$= (x-1)(x^2+3x+2) = (x-1)(x+2)(x+1)$$

So $C_A(x) = (x-1)(x+2)(x+1)$

b) Find the eigenvalues of A (No calculator)

Find roots of $C_A(x)$.

$C_A(x) = 0$

$(x-1)(x+2)(x+1) = 0$

$x-1=0 \quad x+2=0 \quad x+1=0$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$x=1 \quad x=-2 \quad x=-1$

Eigenvalues	Multiplicities
1	1
-2	1
-1	1

(…this is a continuation of problem 3)

Solve $(\lambda I - A)\vec{x} = \vec{0}$

$$\lambda I - A = \begin{bmatrix} \lambda-1 & 15 & 6 \\ 0 & \lambda+4 & 2 \\ 0 & -3 & \lambda-1 \end{bmatrix}.$$

c) Find all eigenvectors of A (Calculator OK)

$\boxed{\lambda=1}$ $(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 0 & 15 & 6 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right] \xRightarrow{RREF} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{array}{l} x_2 = 0 \\ x_3 = 0 \\ x_1 = t \end{array}$

$$E_{\lambda=1} = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R}, t \neq 0 \right\} = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R}, t \neq 0 \right\}$$

$\boxed{\lambda=-2}$ $(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -3 & 15 & 6 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \xRightarrow{RREF} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{array}{l} x_3 = r \\ x_1 + 3x_3 = 0 \Rightarrow x_1 = -3r \\ x_2 + x_3 = 0 \Rightarrow x_2 = -r \end{array}$

$$E_{\lambda=-2} = \left\{ \begin{bmatrix} -3r \\ -r \\ r \end{bmatrix} \middle| \, r \in \mathbb{R}, r \neq 0 \right\} = \left\{ r \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R}, r \neq 0 \right\}$$

$\boxed{\lambda=-1}$ $(\lambda I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & 15 & 6 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right] \xRightarrow{RREF} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\begin{array}{l} x_3 = s \\ x_1 + 2x_3 = 0 \Rightarrow x_1 = -2s \\ x_2 + \frac{2}{3}x_3 = 0 \Rightarrow x_2 = -\frac{2}{3}s \end{array}$

$$E_{\lambda=-1} = \left\{ \begin{bmatrix} -2s \\ -\frac{2}{3}s \\ s \end{bmatrix} \middle| \, \begin{array}{l} s \in \mathbb{R}, \\ s \neq 0 \end{array} \right\} = \left\{ s \begin{bmatrix} -2 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \middle| \, \begin{array}{l} s \in \mathbb{R}, \\ s \neq 0 \end{array} \right\}$$

d) Diagonalize A by finding a diagonalizing matrix P and a diagonal matrix D such that $D = P^{-1}AP$ (No Calculator)

$$P = \begin{bmatrix} 1 & -3 & -2 \\ 0 & -1 & -2/3 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4. (10 points) Find the angle between the vectors $\vec{v} = (4, 2, 9, -2, 3)$ and $\vec{w} = (-3, -2, 1, 2, -5)$

$$\|\vec{v}\| = \sqrt{4^2 + 2^2 + 9^2 + (-2)^2 + 3^2} = \sqrt{114}$$

$$\|\vec{w}\| = \sqrt{(-3)^2 + (-2)^2 + 1^2 + 2^2 + (-5)^2} = \sqrt{43}$$

$$\vec{v} \cdot \vec{w} = (4)(-3) + (2)(-2) + (9)(1) + (-2)(2) + (3)(-5) = -12 - 4 + 9 - 4 - 15 = -26$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$$

$$= \cos^{-1}\left(\frac{-26}{\sqrt{114}\sqrt{43}}\right)$$

$$= \boxed{111.799^\circ}$$

5. (10 points) Find 2 different linear combinations of the vectors $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$.

Lin. combo 1

$$2\vec{v} - 3\vec{w} = 2\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 6 \\ -3 \\ -15 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 3 \\ 23 \end{bmatrix}}$$

Lin. Combo 2

$$0\vec{v} + 4\vec{w} = 0\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \\ -20 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -20 \end{bmatrix}$$

6. (16 points) Prove that the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2y - 3x, 4x)$ is a linear transformation.

Let $\vec{v} = (a,b)$, $\vec{w} = (c,d)$ and let K be a scalar.

① $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$?

$LHS = T(\vec{v}+\vec{w}) = T((a,b)+(c,d)) = T(a+c,\ b+d) = (2(b+d)-3(a+c),\ 4(a+c))$

$= (2b+2d-3a-3c,\ 4a+4c)$

$RHS = T(\vec{v}) + T(\vec{w}) = T(a,b) + T(c,d) = (2b-3a, 4a) + (2d-3c, 4c)$

$= (2b-3a+2d-3c,\ 4a+4c) = (2b+2d-3a-3c,\ 4a+4c)$ Same

So $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ ✓

② $T(K\vec{v}) = KT(\vec{v})$?

$LHS = T(K\vec{v}) = T(K(a,b)) = T(Ka, Kb) = (2Kb-3Ka,\ 4Ka)$ Same

$RHS = KT(\vec{v}) = KT(a,b) = K(2b-3a, 4a) = (K(2b-3a), K(4a)) = (2Kb-3Ka, 4Ka)$

So $T(K\vec{v}) = K\,T(\vec{v})$.

Since $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$ and $T(K\vec{v}) = KT(\vec{v})$, T is a linear Transformation.

7. (12 points) Prove that the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + 2,\ y - 3x)$ is NOT a linear transformation.

T is not a linear transformation, bec. if it was, $T(K\vec{v}) = KT(\vec{v})$ for any scalar K and any vector $\vec{v} \in \mathbb{R}^2$.

Counterexample: Let $K = 2$ and $\vec{v} = (0,1)$.

$T(K\vec{v}) = T(2(0,1)) = T(0,2) = (0+2, 2-3\cdot 0) = (2,2)$ Not the same

$KT(\vec{v}) = 2T(0,1) = 2(0+2, 1-3\cdot 0) = 2(2,1) = (4,2)$

Since $T(K\vec{v})$ is not <u>ALWAYS</u> equal to $KT(\vec{v})$,

T is not a linear transformation.

8. (12, 12, 12 points) Prove or disprove each of the following:

a) If A and B are both $n \times n$ invertible matrices, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

proof: Suppose A and B are invertible. Then A^{-1} and B^{-1} exists.

Let $D = B^{-1}A^{-1}$. I claim D is the inverse of AB.

$$(AB)(D) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(D)(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

So AB is invertible and its inverse is $D = B^{-1}A^{-1}$.

So $(AB)^{-1} = D = B^{-1}A^{-1}$.

b) If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^2$ and k is a scalar, then $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

proof: Let $\vec{u} = (a,b)$, $\vec{v} = (c,d)$, and let k be a scalar.

$$LHS = k(\vec{u}+\vec{v}) = k((a,b)+(c,d)) = k(a+c, b+d) = (k(a+c), k(b+d))$$
$$= (ka+kc, kb+kd)$$

$$RHS = k\vec{u}+k\vec{v} = k(a,b)+k(c,d) = (ka,kb)+(kc,kd) = (ka+kc, kb+kd)$$

Same

So $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$. ✓

c) If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, then $||\vec{v} + \vec{w}|| = ||\vec{v}|| + ||\vec{w}||$

Counterexample: Let $\vec{v} = (1,0,0)$ and $\vec{w} = (-1,0,0)$

$$LHS = ||\vec{v}+\vec{w}|| = ||(1,0,0)+(-1,0,0)|| = ||(0,0,0)|| = \sqrt{0^2+0^2+0^2} = \sqrt{0} = 0$$

Not Same

$$RHS = ||\vec{v}|| + ||\vec{w}|| = ||(1,0,0)|| + ||(-1,0,0)|| = \sqrt{1^2+0^2+0^2} + \sqrt{(-1)^2+0^2+0^2} = 1+1 = 2$$